

# BUNDLES OF $C^*$ -CORRESPONDENCES OVER DIRECTED GRAPHS AND A THEOREM OF IONESCU

JOHN QUIGG

ABSTRACT. We give a short proof of a recent theorem of Ionescu which shows that the Cuntz-Pimsner  $C^*$ -algebra of a certain correspondence associated to a Mauldin-Williams graph is isomorphic to the graph algebra.

## 1. INTRODUCTION

In recent decades the study of fractal geometry has led to the introduction of *graph-directed iterated function systems* [7], also known as *Mauldin-Williams graphs* [2, 3]. These are finite directed graphs of contractions among compact metric spaces. Recently, Ionescu [3] associated a  $C^*$ -correspondence to a given Mauldin-Williams graph, and proved that the resulting Cuntz-Pimsner algebra is isomorphic to the graph  $C^*$ -algebra. Ionescu constructs the isomorphism directly, extending the contractions to the state spaces using Rieffels's theory of Lipschitz metrics on state spaces [9]. Ionescu's result is perhaps surprising, and illustrates important connections among fractal geometry,  $C^*$ -correspondences, and graph algebras. Thus, we feel it will be useful to show how Ionescu's theorem can be quickly deduced from the elementary theory of graph algebras.

We relax Ionescu's hypotheses somewhat: whereas the directed graphs in [3] are finite and have neither sources nor sinks, we only require the graph to be row-finite with no sources. In fact, we only impose these conditions to illustrate our method in its simplest form; the general case could be handled with somewhat more effort.

Also, instead of contractions among metric spaces, we only require continuous maps among locally compact Hausdorff spaces, together with an equivariant surjection from the infinite path space (see Section 3 for details). For finite graphs, such continuous maps together with an equivariant surjection constitute a *self-similarity structure* [6].

---

*Date:* February 2, 2008.

*2000 Mathematics Subject Classification.* Primary 46L08.

*Key words and phrases.* directed graph,  $C^*$ -correspondence, graph  $C^*$ -algebra, Cuntz-Pimsner algebra.

As pointed out in [3, 6], every Mauldin-Williams graph gives rise to a self-similarity structure.

## 2. PRELIMINARIES

Let  $E = (E^0, E^1, r, s)$  be a (directed) graph, with vertices  $E^0$ , edges  $E^1$ , and range and source maps  $r$  and  $s$ . For  $u, v \in E^0$  put  $E_{uv}^1 = \{e \in E^1 \mid r(e) = u, s(e) = v\}$ . Warning: we use the relatively new convention (see [4, 8]) regarding the graph algebra  $C^*(E)$ : the generators go the same way as the edges. Thus, for example, if  $e \in E_{uv}^1$  then  $s_e^* s_e = p_v$  and  $s_e s_e^* \leq p_u$ , and a finite path  $e_1 \cdots e_n$  in  $E$  satisfies  $s(e_i) = r(e_{i+1})$  for  $i = 1, \dots, n-1$ .

For simplicity we assume throughout that  $E$  is row-finite and has no sources, meaning that each vertex receives a positive but finite number of edges.

Let  $E^*$  be the set of finite paths, where vertices are regarded as paths of length 0. Let  $E^\infty$  denote the set of infinite paths, which under our hypotheses is locally compact Hausdorff when given the product topology. Extend the source and range maps to paths in the obvious way, and for  $v \in E^0$  put  $E_v^* = \{\alpha \in E^* \mid r(\alpha) = v\}$ , and similarly for  $E_v^\infty$ .

For  $\alpha \in E^*$  let  $p_\alpha = s_\alpha s_\alpha^*$  be the range projection of the generator  $s_\alpha$ . Put  $A_E = \overline{\text{span}}\{p_\alpha \mid \alpha \in E^*\}$ , a commutative  $C^*$ -subalgebra of  $C^*(E)$ . It is folklore that there is an isomorphism  $\theta: C_0(E^\infty) \rightarrow A_E$  which takes the characteristic function of the set of infinite paths starting with a finite path  $\alpha$  to the generating projection  $p_\alpha$ . We have  $\theta(C_0(E_v^\infty)) = p_v A_E$ .

Each  $e \in E_{uv}^1$  gives rise to a continuous map  $\phi_e^E: E_v^\infty \rightarrow E_u^\infty$  via  $\phi_e^E(\alpha) = e\alpha$ . For  $f \in C_0(E_v^\infty)$  we have

$$\theta(f \circ \phi_e^E) = s_e^* \theta(f) s_e.$$

## 3. IONESCU'S THEOREM

Suppose that for each  $v \in E^0$  we have a  $C^*$ -algebra  $A_v$ , and for each  $e \in E_{uv}^1$  we have an  $A_u - A_v$  correspondence  $X_e$ . Let  $A = \bigoplus_{v \in E^0} A_v$  be the  $c_0$ -direct sum. Then each  $X_e$  can be regarded as a correspondence over  $A$ ; let  $X = \bigoplus_{e \in E^1} X_e$  be the direct sum of these correspondences, with operations

$$(a\xi)_e = a_{r(e)}\xi_e, \quad (\xi a)_e = \xi_e a_{s(e)}, \quad \langle \xi, \eta \rangle_v = \sum_{s(e)=v} \langle \xi_e, \eta_e \rangle$$

for  $a = (a_v)_{v \in E^0} \in A$  and  $\xi = (\xi_e)_{e \in E^1}, \eta = (\eta_e)_{e \in E^1} \in X$ . Then  $X$  is a correspondence over  $A$ . Let  $\mathcal{O}_X$  be the associated Cuntz-Pimsner algebra.

Here we are interested in correspondences arising as follows: for each  $v \in E^0$  let  $T_v$  be a locally compact Hausdorff space, and put  $A_v = C_0(T_v)$ . For each  $e \in E_{uv}^1$  let  $\phi_e: T_v \rightarrow T_u$  be a continuous map, and let  $\phi_e^*: A_u \rightarrow A_v$  be the associated homomorphism, so that  $A_v$  becomes a  $A_u - A_v$  correspondence  $X_e$  with (right) Hilbert module structure coming from the operations of  $A_v$  and left module multiplication defined using  $\phi_e^*$ . Then  $A = \bigoplus_{v \in E^0} A_v$  can be identified with  $C(T)$ , where  $T$  is the disjoint union of  $\{T_v \mid v \in E^0\}$ . Let  $X = \bigoplus_{e \in E^1} X_e$  as above.

If the graph  $E$  is finite, each  $T_v$  is a compact metric space, and each  $\phi_e$  is a contraction, we have a Mauldin-Williams graph, and  $\mathcal{O}_X$  as above was introduced in [1, 3]. Ionescu's proves [3, Theorem 2.3] that  $\mathcal{O}_X$  is isomorphic to the graph algebra  $C^*(E)$ .

Suppose we have a continuous map  $\Phi: E^\infty \rightarrow T$  satisfying  $\Phi \circ \phi_e^E = \phi_e \circ \Phi$  for all  $e \in E^1$ . If  $E$  is finite and  $\Phi$  is surjective, we have a self-similarity structure [6].

Ionescu's theorem follows from the following result:

**Theorem 3.1** ([3, Theorem 2.3]). *With the above notation, suppose that the continuous map  $\Phi: E^\infty \rightarrow T$  is surjective. Then the Cuntz-Pimsner algebra  $\mathcal{O}_X$  is isomorphic to the graph algebra  $C^*(E)$ .*

*Proof.* Define  $\pi: C_0(T) \rightarrow C^*(E)$  by  $\pi(f) = \theta(f \circ \Phi)$ , and for each  $e \in E^1$  define  $\psi_e: X_e \rightarrow C^*(E)$  by  $\psi_e(\xi) = s_e \pi(\xi)$ . Routine calculations, using  $p_v = \theta(\chi_{T_v} \circ \Phi)$  and commutativity of  $A_E$ , show that the pair  $(\psi_e, \pi)$  is a covariant representation of the correspondence  $X_e$ . We can form the direct sum  $\psi = \bigoplus_{e \in E^1} \psi_e$ , giving a Cuntz-Krieger Toeplitz representation  $(\psi, \pi)$  of the correspondence  $X$  in  $C^*(E)$ . Since the range of  $(\psi, \pi)$  contains the generators of  $C^*(E)$ , an application of the Gauge-Invariant Uniqueness Theorem [5, Theorem 6.4] shows that the associated homomorphism  $\psi \times \pi: \mathcal{O}_X \rightarrow C^*(E)$  is an isomorphism.  $\square$

## REFERENCES

- [1] R. Bartholdi, R. Grigorchuk, and V. Nekrashevych, *From fractal groups to fractal sets*, Trends in Mathematics: Fractals in Braz 2001, Birkhauser, 2003, pp. 25–18.
- [2] G. A. Edgar, *Measure, topology, and fractal geometry*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1990.
- [3] M. Ionescu, *Operator Algebras and Mauldin Williams Graphs*, Preprint (2004).

- [4] T. Katsura, *A class of  $C^*$ -algebras generalizing both graph algebras and homeomorphism  $C^*$ -algebras, I. Fundamental results*, Trans. Amer. Math. Soc. **356** (2004), 4287–4322.
- [5] ———, *On  $C^*$ -algebras associated with  $C^*$ -correspondences*, J. Funct. Anal. **217** (2004), 366–401.
- [6] J. Kigami, *Analysis on Fractals*, Cambridge University Press, 2001.
- [7] R. D. Mauldin and S. C. Williams, *Hausdorff dimension in graph directed constructions*, Trans. Amer. Math. Soc. **309** (1988), 811–829.
- [8] I. Raeburn, *Graph algebras*, CBMS Reg. Conf. Ser. Math., American Mathematical Society, Providence, RI, 2005.
- [9] M. A. Rieffel, *Matrices on state spaces*, Doc. Math. **4** (1999), 559–600.

DEPARTMENT OF MATHEMATICS AND STATISTICS, ARIZONA STATE UNIVERSITY,  
TEMPE, ARIZONA 85287

*E-mail address:* quigg@math.asu.edu